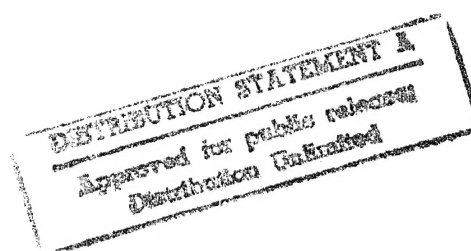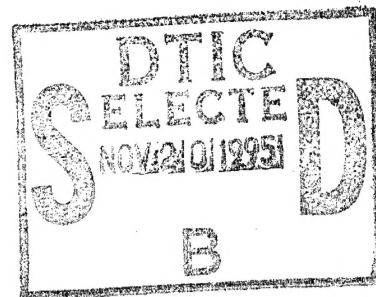NASA Contractor Report 3977

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STRESS WAVES IN TRANSVERSELY ISOTROPIC MEDIA —

THE HOMOGENEOUS PROBLEM

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SUMMARY

The homogeneous problem of stress wave propagation in unbounded transversely isotropic media is analyzed. By adopting plane wave solutions, the conditions for the existence of the solution are established in terms of phase velocities and directions of particle displacements. Dispersion relations and group velocities are derived from the phase velocity expressions. The deviation angles - angles between the normals to the adopted plane waves and the actual directions of their propagation - are numerically determined for a specific fiberglass/epoxy composite. A graphical method for the construction of the wave surfaces is introduced, using magnitudes of phase velocities and deviation angles. The simpler results for the case of isotropic media are shown to be contained in the solutions for the transversely isotropic media.

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INTRODUCTION

Most studies on of stress waves in solid elastic materials refer to isotropic media. The usual model adopted to describe the geometry of such waves is based on the theory of sound waves in homogeneous materials as first presented by Lord Rayleigh [1]. The same concepts of wave front, dispersion relations and directivity functions which are used to describe sound waves are also used in the description of stress waves in isotropic solids, as can be seen, for instance, in the work of Miller and Pursey [2] and Achenbach [3]. Because an isotropic medium has the simplest dynamic behavior among all possible cases of symmetry, it can be used as a "reference" when anisotropic cases are studied. Concepts (such as phase and group velocities, velocity surfaces and wave surfaces) used to describe the propagation of waves in isotropic media have also been applied to anisotropic media [4-8].

This work is focused on the problem of propagation in anisotropic media. The behavior of stress waves in one type of anisotropic medium, the transversely isotropic material, is described for an infinite medium, establishing the parallel between such materials and the isotropic case. Also a graphical technique for the construction of the wave surface is introduced, giving a better understanding of special geometric features that frequently occur during the propagation in filamentary materials.

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BASIC CONCEPTS

Definition of Phase Velocity

The definition of phase velocity is formulated by following the path of a small segment of a wave front propagating through a medium . The segment is assumed to be sufficiently small that it can be approximated by a plane wave segment. The concept is valid for any elastic medium ,isotropic or anisotropic.

The phase velocity is defined for periodic or nonperiodic waves as [9]

$$v_n = d / t \qquad (1)$$

where d is the distance travelled by the plane segment in the direction of the normal $\underline{n}$ to the wave front in time t (see Figs. 1.a and 1.b) . The subscript n for the velocity indicates that the phase velocity represents the speed of propagation of the front segment in the direction of the normal $\underline{n}$.

For periodic waves, the phase velocity can be represented in terms of either wavelength and period or frequency and wave number as [8]

$$v_n = \lambda / T = \omega / k \qquad (2)$$

where λ is the wavelength , T is the period of the wave, ω is the radian frequency and k is the wave number magnitude.

It is assumed that, in general, the plane wave segment travels along a direction that is different from the the wave front normal $\underline{n}$. If this direction is called $\underline{r}$ (as for ray of

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propagation),the angle between r and n is defined as the deviation angle [10]. It is known that the deviation angle for isotropic materials is zero [9] .

Also, it can be shown that stress waves in infinite elastic media are nondispersive [4,6] . As a consequence , the phase velocities and deviation angles are independent of frequency for any symmetry configuration of properties of the medium , including the isotropic and transversely isotropic media considered here .

Definition of Group Velocity

The group velocity is defined as the velocity of propagation of energy in a medium [4].

For a periodic wave propagating in a medium, the components v_{gi} of the group velocity in a cartesian coordinate system x, y and z are [9]

$$v_{gi} = d\omega/dk \qquad (3)$$

where the k_i's (i=x,y,z) are the components of the wave number vector along the coordinate axes and ω represents the radian frequency of oscillation of the wave . The frequency is usually represented as a function of the wave number components and the elastic properties of the medium ; this representation is known as the dispersion relation [11]. If the dispersion relation is known, the derivatives in eqn.(3) can be easily calculated. These relations will be derived later from the expressions of phase velocity for the media of interest, including isotropic and transversely isotropic materials.

In unbounded elastic isotropic media the elastic properties are the same in all directions. For this reason it can be shown that the magnitudes of the phase and group velocities are equal and the directions n and r are coincident [7] (see Fig.1.a). These results are to be expected since the system is nondispersive.

For a transversely isotropic medium, the magnitudes of phase velocity and group velocity are distinct, and it can be shown that the deviation angle $\triangle$,between n and r,has the value [10]

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$$\cos \triangle = v_n / v_g \tag{4}$$

where $v_g = (v_{gx}^2 + v_{gy}^2 + v_{gz}^2)^{1/2}$. The deviation angle $\triangle$ is indicated in Fig. 1b .

The physical meaning of a nonzero deviation angle is that energy propagates obliquely with respect to the wave front normal. The fact that nondispersive systems can have distinct values of phase and group velocities was first emphasized by Lighthill [12].

PLANE WAVE SOLUTION OF THE HOMOGENEOUS PROBLEM

The solution of the equations of motion is derived for the specific problem of the propagation of stress waves in transversely isotropic media. All the assumptions and results also apply to isotropic media ,as they are a subclass of transversely isotropic media . In the following sections all results are first derived for a transversely isotropic medium and then simplified to an isotropic medium.

Equations of Motion for Transversely Isotropic Medium

The following assumptions are made for the derivation of the equations of motion:

(1) The medium is homogeneous and has constant density.

(2) The medium obeys Hooke's law.

The equations of motion can be obtained for any elastic medium by the force-dynamic requirements of a volumetric element. Using a cartesian coordinate system, oxyz, as the reference system, these equations can be written as [9]

$$\tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z} = \rho u_{,tt}$$

$$\tau_{xy,x} + \tau_{yy,y} + \tau_{yz,z} = \rho v_{,tt} \qquad (5)$$

$$\tau_{xz,x} + \tau_{yz,y} + \tau_{zz,z} = \rho w_{,tt}$$

where τ_{rs} (r,s=x,y,z) are the normal (r=s) and shear (r≠s) stresses with respect to system oxyz;

u,v and w are the displacement components of a point in

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the medium along the directions x,y and z, respectively; and

ρ is the density.

The indexes following commas denote derivatives.

Now, assume a general orthotropic medium in which the principal axes coincide with the reference system oxyz. The generalized Hooke's law for this medium is [13]

$$\tau_{xx} = C_{11}u_{,x} + C_{12}v_{,y} + C_{13}w_{,z}$$

$$\tau_{yy} = C_{12}u_{,x} + C_{22}v_{,y} + C_{23}w_{,z}$$

$$\tau_{zz} = C_{13}u_{,x} + C_{23}v_{,y} + C_{33}w_{,z} \tag{6}$$

$$\tau_{xz} = C_{44}(u_{,z} + w_{,x})$$

$$\tau_{yz} = C_{55}(v_{,z} + w_{,y})$$

$$\tau_{xy} = C_{66}(u_{,y} + v_{,x})$$

where the C_{ij} are the nine independent elastic constants of the stifness matrix.

Here the interest is focused on transversely isotropic media, which comprise a subclass of the orthotropic media described by eqn.(6), where the number of independent constants C_{ij} is reduced to five by the following constraints:

$$C_{11} = C_{22}, \quad C_{13} = C_{23}, \quad C_{44} = C_{55} \quad \text{and} \quad C_{66} = (C_{11} - C_{12})/2 \tag{7}$$

In accordance with eqn.(7), the xy plane is taken to be the isotropic plane for elastic properties.

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The equations of motion, eqn.(5), can be written in terms of the displacements u, v and w ,using eqns.(6) and (7) for the transversely isotropic medium ,as

$$C_{11}u_{,xx}+C_{12}v_{,yx}+C_{13}w_{,zx}+C_{66}(u_{,yy}+v_{,xy})+C_{44}(u_{,zz}+w_{,xz})=\rho u_{,tt}$$

$$C_{66}(u_{,yx}+v_{,xx})+C_{12}u_{,xy}+C_{11}v_{,yy}+C_{13}w_{,zy}+C_{44}(v_{,zz}+w_{,yz})=\rho v_{,tt} \quad (8)$$

$$C_{44}(u_{,zx}+w_{,xx})+C_{44}(v_{,zy}+w_{,yy})+C_{13}(u_{,xz}+v_{yz})+C_{33}w_{,zz}=\rho w_{,tt}$$

Conditions for Existence of Plane Wave Solution

Now, assume a plane wave solution of the form [10]

$$(u,v,w)=(P_x,P_y,P_z)\ \exp[\frac{2\pi i}{\lambda}(xn_x+yn_y+zn_z-v_n t)] \quad (9)$$

where P_x, P_y, P_z are the amplitude components of a particle displacement vector along the coordinate axes x, y and z, respectively, corresponding to a plane wave with unit normal $\underline{n}$; n_x, n_y and n_z are the direction cosines of the unit normal $\underline{n}$ along the coordinate axes x, y and z, respectively; v_n is the phase velocity defined for the direction $\underline{n}$, and λ is the associated wavelength with the selected n and v_n.

If the assumed solution , eqn.(9), is substituted into the equations of motion, eqn.(8), the following expressions are obtained [10]:

$$[C_{11}n_x^2+C_{66}n_y^2+C_{44}n_z^2-\rho v_n^2]P_x+(C_{12}+C_{66})n_xn_yP_y+(C_{13}+C_{44})n_xn_zP_z=0$$

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$$(C_{12}+C_{66})n_x n_y P_x + [C_{66}n_x^2 + C_{11}n_y^2 + C_{44}n_z^2 - \rho v_n^2]P_y + (C_{13}+C_{44})n_y n_z = 0$$

$$(C_{13}+C_{44})n_x n_z P_x + (C_{13}C_{44})n_y n_z P_y + [C_{44}(n_x^2+n_y^2)+C_{33}n_z^2 - \rho v_n^2]P_z = 0$$

(10)

The condition for the existence of the plane wave solution can be expressed by setting the determinant of the matrix of the coefficients of P_x, P_y and P_z in eqn.(10) equal to zero [10,14]:

$$\begin{vmatrix} [C_{11}n_x^2+C_{66}n_y^2+C_{44}n_z^2-\rho v_n^2] & (C_{12}+C_{66})n_x n_y & (C_{13}+C_{44})n_x n_z \\ (C_{12}+C_{66})n_x n_y & [C_{66}n_x^2+C_{11}n_y^2+C_{44}n_z^2-\rho v_n^2] & (C_{13}+C_{44})n_y n_z \\ (C_{13}+C_{44})n_x n_z & (C_{13}+C_{44})n_y n_z & [C_{44}(n_x^2+n_y^2)+C_{33}n_z^2-\rho v_n^2] \end{vmatrix} = 0$$

(11)

Eqn.(11) is known as Christoffel's equation [5] and is written here specifically for transversely isotropic materials, considering the symmetry conditions and relations between the elastic constants of the stiffness matrix given in eqn.(7). The expression for the determinant is a cubic equation in v_n, and its solution gives three possible values of v_n for each selected set of n_x, n_y and n_z. (Recall that each unit normal vector $\underline{n}$ in the medium is uniquely associated with a set of components n_x, n_y and n_z.) Moreover, the elastic properties are symmetric with respect to the z axis, so the direction cosines for the normal $\underline{n}$ can be expressed in terms of the angle between $\underline{n}$ and the z axis, as [14]

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$$n_x^2 + n_y^2 = \sin^2\theta \qquad\qquad (12)$$
$$n_z^2 = \cos^2\theta$$

The advantage of expressing the direction cosines in terms of the angle θ is that the expressions for the velocities can be written in terms of a single variable.

Expressions for Phase Velocities

By combining eqns.(11) and (12), the expressions for the phase velocities associated with the possible plane wave solutions can be written as follows:

-For a transversely isotropic medium

$$(v_n)_I = [C_{66}\sin^2\theta + C_{44}\cos^2\theta)/\rho]^{1/2}$$

$$(v_n)_{II} = [(C_{44} + C_{11}\sin^2\theta + C_{33}\cos^2\theta - \sqrt{\varepsilon})/2\rho]^{1/2} \qquad (13)$$

$$(v_n)_{III} = [(C_{44} + C_{11}\sin^2\theta + C_{33}\cos^2\theta + \sqrt{\varepsilon})/2\rho]^{1/2}$$

where

$$\varepsilon = [(C_{11} - C_{44}\sin^2\theta + (C_{44} - C_{33})\cos^2\theta]^2 + 4(C_{13} + C_{44})^2(\sin\theta\cos\theta)^2$$

-For an isotropic medium

$$(v_n)_I = [C_{66}/\rho]^{1/2}$$

$$(v_n)_{II} = [C_{66}/\rho]^{1/2} \qquad (14)$$

$$(v_n)_{III} = [C_{11}/\rho]^{1/2}$$

Where for the isotropic medium the following constraint relations between the elastic constants of the stiffness matrix have been applied:

$$C_{11}=C_{22}=C_{33}, \qquad C_{12}=C_{13}=C_{23}, \qquad C_{44}=C_{55}=C_{66}=(C_{11}-C_{12})/2 \qquad (15)$$

The isotropic medium results may be used as a means of checking the more general results for the transversely isotropic medium. The results in eqn.(14) are derived from the corresponding velocity expressions for the transversely isotropic medium. Observe that the values obtained for the phase velocities in eqn.(14) are the widely known values of velocities of shear waves, $(v_n)_I$ and $(v_n)_{II}$, and longitudinal waves, $(v_n)_{III}$, which in terms of elastic properties can also be represented by $(G/\rho)^{1/2}$ and $(E/\rho)^{1/2}$, respectively. The values G and E are the shear and extensional elastic moduli, repectively.

Particle Displacement Vector Components

By using the computed values of the phase velocities , the amplitudes of the particle displacement components associated with each of the phase velocity vectors can be found . This can be achieved by solving eqn.(10) for the displacement amplitude components P_x, P_y and P_z for each of the phase velocities. Because at each point there are three velocities for each direction defined by $\underline{n}$, there are also three displacement amplitude vectors for each direction $\underline{n}$, namely, $(\underline{P})_I=(P_x,P_y,P_z)_I$, $(\underline{P})_{II}=(P_x,P_y,P_z)_{II}$ and $(\underline{P})_{III}=(P_x,P_y,P_z)_{III}$ corresponding to

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$(v_n)_I$, $(v_n)_{II}$ and $(v_n)_{III}$, respectively. The three displacement vectors are normal to each other but usually neither normal to nor coincident with the direction $\underline{n}$ for a transversely isotropic medium.

In the homogeneous problem, there is no unique solution for the displacement amplitude components. Each phase velocity $(v_n)_I$, $(v_n)_{II}$ and $(v_n)_{III}$ has a corresponding displacement vector $(\underline{P})_I$, $(\underline{P})_{II}$ and $(\underline{P})_{III}$, respectively, which represents a mode of vibration. Any multiple of the three displacement vectors $(\underline{P})_I$, $(\underline{P})_{II}$ and $(\underline{P})_{III}$ is also a solution.

As there are no initial conditions in the homogeneous problem, the components P_x, P_y and P_z obtained by solving eqn.(10) for each $(v_n)_i$ can be expressed in terms of an arbitrary constant. Thus, the useful information that can be extracted from the solution of eqn.(10) for the displacements consists of the direction of the particle displacement vector with respect to the coordinate axes. The magnitude of the displacement vectors $(P)_i$ is expressed in terms of the components as

$$(P)_i = \left[(P_x^2 + P_y^2 + P_z^2)^{1/2} \right]_i \qquad\qquad i=I,II,III \qquad\qquad (16)$$

Then, the direction cosines of the displacement vectors with respect to the xyz coordinate system are [10]

$$\frac{P_x}{P} = \frac{[n_x^2(C_{12}+C_{66})]^{1/2}}{B\ [\rho v_n^2 - (n_x^2+n_y^2)C_{66} - n_z^2 C_{44}]}$$

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$$\frac{P_y}{P} = \frac{[n_y^2(C_{12}+C_{66})]^{1/2}}{B\,[\,\rho v_n^2 - (n_x^2+n_y^2)C_{66} - n_z^2 C_{44}\,]} \qquad (17)$$

$$\frac{P_z}{P} = \frac{[n_z^2(C_{13}+C_{44})^2/(C_{12}+C_{66})]^{1/2}}{B\,\left[\,\rho v_n^2 - (n_x^2+n_y^2)C_{44} - n_z^2 C_{33} + n_z^2 \dfrac{(C_{13}+C_{44})^2}{(C_{12}+C_{66})}\,\right]}$$

where

$$B = \left\{ \frac{(n_x^2+n_y^2)\,(C_{12}+C_{66})}{[\,\rho v_n^2 - (n_x^2+n_y^2)C_{66} - n_z^2 C_{44}\,]^2} \right. +$$

$$\left. \frac{[n_z^2(C_{13}+C_{44})^2/(C_{12}+C_{66})]}{\left[\,\rho v_n^2 - (n_x^2+n_y^2)C_{44} - n_z^2 C_{33} + n_z^2 \dfrac{(C_{13}+C_{44})^2}{(C_{12}+C_{66})}\,\right]^2} \cdot 2 \right\}^{1/2}$$

Observe from eqn.(17) that the sum of the squares of the ratios P_x/P, P_y/P and P_z/P is equal to one, meaning that only two of the three components can be independently determined.

The isotropic medium is a particular case of the transversely isotropic medium and where the particle displacement directions are either normal to or parallel with the direction n. In this case, the displacement vectors are aligned with or perpendicular to the direction n.

VALUES OF ELASTIC CONSTANTS OF THE STIFFNESS MATRIX

In order to demonstrate the application of the plane wave solution for the homogeneous problem of wave propagation in infinite media, numerical values are fixed for the elastic constants of the stiffness matrix, representing the two media of interest: transversely isotropic and isotropic materials.

Transversely Isotropic Material

For the transversely isotropic material, there are five independent elastic constants in the stiffnes matrix that will be taken as C_{11}, C_{12}, C_{13}, C_{33} and C_{44}. Note that $C_{12} = C_{11}-2C_{66}$. In this case, the material is a unidirectional fiberglass epoxy composite having a resin content of 36 percent by weight and a density of 1850 kg/m^3. The elastic constants of the stiffness matrix were determined experimentally by ultrasonic techniques from samples of the material as [15]

$$C_{11} = 10.581 \times 10^9 \ N/m^2$$
$$C_{13} = 4.679 \times 10^9 \ N/m^2$$
$$C_{33} = 40.741 \times 10^9 \ N/m^2$$
$$C_{44} = 4.422 \times 10^9 \ N/m^2$$
$$C_{66} = 3.243 \times 10^9 \ N/m^2$$

The constant C_{12} was calculated from the values of C_{11} and C_{66} above as $C_{12} = 4.098 \times 10^9 \ N/m^2$. See Fig.2 for the relation between the coordinate axes and the material's orientation.

Isotropic Material

For an isotropic material, there are two independent elastic constants in the stiffness matrix that will be taken as C_{11} and C_{12}. For the representative isotropic material, E glass was chosen.

The values of the corresponding elastic constants are $C_{11} = 82.658 \times 10^9$ N/m^2 and $C_{12} = 23.31 \times 10^9$ N/m^2 with a density of 2540 kg/m^3 [16]. This material in fiber form is one of the constituents of the transversely isotropic material described above.

POLAR DIAGRAMS OF PHASE VELOCITIES

The values of the phase velocities obtained from eqns.(13) and (14), using the numerical values of the material properties, can be presented in polar diagrams. Such polar diagrams of phase velocity are called velocity surfaces [5].

The velocity surfaces are shown in Figs. 3 and 4 for the isotropic material and the transversely isotropic material,respectively. One quarter of each surfaces' intersection with the xz plane is presented . The surfaces are designated as V(SH), V(SV) and V(P) corresponding to the velocities indexed as I, II and III, respectively.

Among the directions defined by the normal n used to calculate the points of the velocity surfaces, several directions were selected and presented in. Figs. 3 and 4. These directions are represented as the radii from the origin of the coordinate system to the surfaces. The particle displacement vectors corresponding to these directions are represented by the arrows and dots , giving an indication of the relative position between the particle displacement direction and the direction of the normal to the wave front n. The relative position of the particle displacement direction and the corresponding direction of the normal n (radii from the origin) can be used to identify the modes of 'propagation. If these two directions are coincident, the mode is purely longitudinal. If the two directions are perpendicular ,the mode is purely transverse.

Observation of the V(P) surface in Fig. 3 shows that directions n and particle displacement directions are coincident

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for all directions in the medium, indicating the existence of a pure longitudinal mode of propagation . The surfaces V(SH) and V(SV) in Fig.3 show that directions n and particle displacement directions are normal to each other for all directions in the medium, indicating the existence of two pure transverse modes of propagation. So, for isotropic materials there are three possible pure modes of propagation, one longitudinal and two transverse.

For the transversely isotropic material, there is one possible pure transverse mode propagating in all directions of the medium, corresponding to the V(SH) surface as can be seen in Fig.4. Pure longitudinal modes occur only at the intersections of the V(P) surface with the coordinate axes (x, y and z) where the particle displacement direction coincides with the direction n. For the V(SV) surface there are three directions where pure transverse modes occur, namely, the directions of the coordinate axes x, y and z. (In Fig.4 only the intersections with the x and z axes are shown).

According to the convention adopted for the designation of the surfaces, Table 1 shows the modes of vibration for points along the directions of the coordinate axes and the surfaces that contain the indicated modes . Other designations for the surfaces can be found in the literature [10,14,17], as, for instance, "transverse" for the SH waves, "quasi-transverse" for the SV waves, and "quasi-longitudinal" for the P waves .

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DISPERSION RELATIONS AND GROUP VELOCITIES

After the phase velocities for each direction $\underline{n}$ are obtained, the group velocities can be determined by the use of the dispersion relation. Combining the definition of phase velocity, eqn.(2), and the expressions obtained for the velocities, eqns.(13) and (14), the dispersion relations can be written as follows:

-For a transversely isotropic medium

$$(\omega)_I = \{[C_{66}(k_x^2+k_y^2)+C_{44}k_z^2]/ \rho \}^{1/2}$$

$$(\omega)_{II} = \{[C_{44}k^2+C_{11}(k_x^2+k_y^2)+C_{33}k_z^2-\sqrt{\varepsilon k^4}]/2 \rho \}^{1/2} \qquad (18)$$

$$(\omega)_{III} = \{[C_{44}k^2+C_{11}(k_x^2+k_y^2)+C_{33}k_z^2+\sqrt{\varepsilon k^4}]/2 \rho \}^{1/2}$$

-For an isotropic medium

$$(\omega)_I = [C_{66}(k_x^2+k_y^2+k_z^2)/ \rho]^{1/2}$$

$$(\omega)_{II} = [C_{66}(k_x^2+k_y^2+k_z^2)/ \rho]^{1/2} \qquad (19)$$

$$(\omega)_{III} = [C_{11}(k_x^2+k_y^2+k_z^2)/ \rho]^{1/2}$$

where the square of the wave number magnitude is represented by

$$k^2= (k_x^2+k_y^2+k_z^2) \qquad (20)$$

and the relations between the components of $\underline{k}$, namely,

$$k_x^2+ k_y^2 = k^2\sin^2\theta, \qquad k_z^2 = k^2\cos^2\theta \qquad (21)$$

were used (as in eqn.(12) for $\underline{n}$).

From the dispersion relations in eqns.(18) and (19) and the definition in eqn.(3), the values of the group velocity components along the coordinate axes can be determined . The expressions for the group velocity components in the xz plane are as follows :

-For a transversely isotropic medium

$$(v_{gx})_I = n_x C_{66} [(C_{66}n_x^2 + C_{44}n_z^2) \rho]^{-1/2}$$

$$(v_{gz})_I = n_z C_{44} [(C_{66}n_x^2 + C_{44}n_z^2) \rho]^{-1/2}$$

$$(v_{gx})_{II} = (1/B)\{n_x(C_{44}+C_{11}) - \varepsilon^{-1/2}n_z(C_{11}-C_{44})[(C_{11}-C_{44})n_x^2 +$$
$$(C_{44}-C_{33})n_z^2] + 2n_x n_z^2 (C_{13}+C_{44})^2\}$$

(22)

$$(v_{gz})_{II} = (1/B)\{n_z(C_{44}+C_{33}) - \varepsilon^{-1/2}n_z(C_{44}-C_{33})[(C_{11}-C_{44})n_x^2 +$$
$$(C_{44}-C_{33})n_z^2] + 2n_z n_x^2 (C_{13}+C_{44})^2\}$$

$$(v_{gx})_{III} = (1/C)\{n_x(C_{44}+C_{11}) + \varepsilon^{-1/2}n_x(C_{11}-C_{44})[(C_{11}-C_{44})n_x^2 +$$
$$(C_{44}-C_{33})n_z^2] + 2n_x n_z^2 (C_{13}+C_{44})^2\}$$

$$(v_{gz})_{III} = (1/C)\{n_z(C_{44}+C_{33}) + \varepsilon^{-1/2}n_z(C_{44}-C_{33})[(C_{11}-C_{44})n_x^2 +$$
$$(C_{44}-C_{33})n_z^2] + 2n_z n_x^2 (C_{13}+C_{44})^2\}$$

where $B = \{[C_{44}(n_x^2 + n_z^2) + C_{11}n_x^2 + C_{33}n_z^2 - \sqrt{\varepsilon}]/2 \rho\}^{1/2}$

$$C = \{[C_{44}(n_x^2 + n_z^2) + C_{11}n_x^2 + C_{33}n_z^2 + \sqrt{\epsilon}] / 2\rho\}^{1/2}$$

-For an isotropic medium

$$(v_{gx})_I = (v_{gx})_{II} = n_x C_{66} [C_{66} \rho (n_x^2 + n_z^2)]^{-1/2}$$

$$(v_{gz})_I = (v_{gz})_{II} = n_z C_{66} [C_{66} \rho (n_x^2 + n_z^2)]^{-1/2}$$

$$(v_{gx})_{III} = n_x C_{11} [C_{11} \rho (n_x^2 + n_z^2)]^{-1/2}$$

$$(v_{gz})_{III} = n_z C_{11} [C_{11} \rho (n_x^2 + n_z^2)]^{-1/2}$$

(23)

Observe that the magnitudes of the group velocities can be obtained from the corresponding components in the xz plane as

$$(v_g)_{I,II,III} = [(v_{gx}^2 + v_{gz}^2)^{1/2}]_{I,II,III}$$

(24)

For an isotropic medium, the magnitudes of the group velocities are identical to those of the phase velocities as defined in eqns.(14) .

CALCULATION OF DEVIATION ANGLES

The purpose of calculating the deviation angles is to determine the actual direction $\underline{r}$ of propagation of plane wave segments having normal $\underline{n}$. The deviation angle can be expressed in terms of the phase and group velocity magnitudes as given in eqn.(4).

Because z is the symmetry axis in the transversely isotropic medium under consideration , the directions of phase velocity and group velocity ($\underline{n}$ and $\underline{r}$, respectively) are expressed in terms of their angles with respect to this axis .

For the phase velocity direction , the angle θ with respect to the z axis (see Fig.1b) is defined by the direction cosines of $\underline{n}$ as given in eqn.(12) as

$$(\tan \theta)_{I,II,III} = [(n_x^2 + n_y^2)^{1/2} / n_z]_{I,II,III} \qquad (25)$$

The angle θ' of the direction $\underline{r}$ (for which the group velocity is defined) with respect to the z axis can be expressed in terms of the group velocity components as

$$(\tan \theta')_{I,II,III} = [(v_{gx}^2 + v_{gy}^2)^{1/2} / v_{gx}]_{I,II,III} \qquad (26)$$

The indexes I,II and III in eqns.(25) and (26) denote that the same equations apply to each of the velocity sets. Moreover, due to symmetry, n_y and v_{gy} may be taken as zero ,and the calculation can be done for the xz plane .

The general expression for the deviation angle is then

$$\triangle = \theta' - \theta \qquad (27)$$

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The specific algebraic expressions, expressed in terms of elastic constants for the deviation angles of transversely isotropic media, are not reproduced here. For isotropic media the deviation angles are zero. It is interesting to note that algebraic manipulations show that the deviation angle is independent of the frequency. And, since the phase velocity is also independent of frequency (see eqns.(13) and (14)), so is the group velocity, which is another way of expressing the nondispersive character of the elastic media under study.

The calculated deviation angles for the transversely isotropic medium under consideration are shown in Figs. 5a ,5b and 5c , for the SH,SV and P modes, respectively.

GRAPHIC CONSTRUCTION OF WAVE SURFACES

In this section the graphic construction of the wave front will be described, focusing on the transversely isotropic material. The quantities needed for this construction are the magnitudes of the phase velocities ,the directions $\underline{n}$ for which these velocities are defined and the deviation angles.

In other words , the locus of the points of equal phase in geometric space for unit time will be determined, supposing that plane waves passed through the origin at time t=0 in all possible directions defined by their normals $\underline{n}$. The construction is described for the xz plane but it is valid for any plane containing the symmetry axis z.

The following steps must be followed for the construction (see Fig.(6)):

(1) Choose a direction $\underline{n}$ (n_x, n_z)

(2) Draw a line from the origin of the coordinate system in the direction $\underline{n}$ and having a scaled length equal to v_n. Since time is taken as unity, the segment represents a distance.

(3) From the line with direction $\underline{n}$, mark the deviation angle $\triangle$, if positive below ,if negative above the line having the direction $\underline{n}$. The direction so determined is $\underline{r}$ which is the direction of the the group velocity v_g .

(4) From the tip of the line of length v_n, draw a line perpendicular to it.

(5) The intersection of the perpendicular line with the

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direction $\underline{r}$ determines a point P on the wave surface. Thus, point P represents the location at time t=1 sec of a point of the wave front which at time t=0 passed through the origin with velocity v_n in the direction of $\underline{n}$.

If the construction described is repeated for each set of calculated phase velocities assuming all possible directions $\underline{n}$ of the wave normal in the medium , the wave surfaces are obtained. The construction steps of the wave surfaces W(SV) and W(P) corresponding to velocity surfaces V(SV) and V(P) respectively, are shown in Figs. 7 and 8 for some directions in the representative transversely isotropic medium. In these figures, the primed points are points of the extremities of the segments of length v_n as described in step 2). The umprimed points are the corresponding points on the wave surface.

The construction of the wave surface W(SH) corresponding to velocities V(SH) is not shown since the deviation angles are small and the final shape of W(SH) is very similar to the shape of V(SH). The three wave surfaces obtained for the representative transversely isotropic medium are shown together (positive x-z quadrant only) in Fig. 9 .

For the isotropic medium no construction is made since the deviation angles are zero. For such a case, the wave surfaces are identical to the corresponding velocity surfaces.

It is important to emphasize that the wave surfaces are the actual geometric positions of wave fronts for the three possible modes of propagation. Because plane wave solutions are assumed to generate these surfaces ,the wave surfaces represent the envelope

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of all possible plane waves emanating from the origin of the coordinate system in all possible directions, as if such a source existed at the origin.

SPECIAL FEATURES OBSERVED IN THE WAVE SURFACE W(SV)

As it can be observed in Fig. 7, for the representative transversely isotropic medium there is a folding of the W(SV) wave surface due to an inversion of the sign of the deviation angle .

To explain the geometric shape of the wave surface W(SV), assume that the wave front is constituted of many small segments of plane waves having the same length and passing through the origin at time t=0. Assume that for an infinitesimal interval of time these segments of plane wave are still very close to the origin and form a polygonal line that can be inscribed in an arc of circumference. Then immediately thereafter, the segments start to deviate from their normals, following the directions $\underline{r}$ defined by the deviation angles .If it is assumed that the segments' extremities remain connected physically , the wave front must be continuous in time as these segments will be stretched as they travel away from the origin.

The deviation angles, Fig. 5b, can be associated with a degree of stretching of the original straight segments . The larger the deviation angle ,the larger the degree of stretching undergone by the segment. Recall that the angle θ represents the angle of the normal $\underline{n}$ (to each segment) with respect to the z axis.

Observe that the curve of $(\triangle)_{II}$ has two maxima, one positive at $\theta = 33°$ and one negative at $\theta = 77.5°$, and passes through zero at $\theta = 60°$. If the curve is divided in four regions as shown in Fig. 10b ,it can be observed that in region 1

the segments are stretched increasingly until the first maximum at $\theta = 33°$. Then for region 2 the segments are less and less stretched until the deviation angle reaches the value zero, represented by point C in Fig. 5.a. The segment corresponding to the zero deviation is not stretched. For region 3, again the segments are stretched from zero to the second maximum at $\theta = 77.5°$ and finally in region 4 they are less and and less stretched until reaches zero for the segment whose normal is located at $90°$ with respect to the z axis.

In the model of the polygonal line constituted of straight segments, if points A and E, Fig. 10.a, are the intersections of this line with the coordinate axes, for an infinitesimal time after zero time ,these points remain on the coordinate axes since the corresponding deviation angles are zero. Later ,at a time t equals unity, these points will be at positions A' and E', respectively. Considering that at the same time all segments are supposed to remain connected ,the segments originally in regions 1 and 2 (Fig. 10.b) will undergo an effect of pulling away from the z axis; the segments originally in regions 3 and 4 on the other hand , will undergo an effect of pulling away from the x axis, moving closer toward the z axis.

It has been determined that the segments which are the tips of the cuspidal edges B',D' of the wave surface W(SV) have normals n at angles of $54.02°$ and $75.06°$, respectively, with respect to the z axis. For time t equals unity, the location of these two segments is defined by the lines passing through the origin and the points B' and D', respectively, at $61.86°$ and

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41.93° with respect to the z axis (see Fig. 10a).

The originally equal segments that at time t equals unity are located inside the region limited by the lines OB'and OD' can be associated with the amount of energy that is concentrated in this region. If it is assumed that each segment that passes through the origin at time t=0 carries the same amount of energy, the percentage of the total number of the segments contained in the region of interest at time t equals unity is also the percentage of energy contained in the region at any other time t. Using this concept, it was found numerically that 64.4% of the total energy propagates through the region located between the lines OB' and OD' where the included angle B'OD' is 20.93°. This calculation was made for a total number of 180 segments.

Concerning the geometry of propagation of the entire wave front, it can be said that if an observer stands at a position with respect to the coordinate axes inside the region defined by lines OB' and OD' (for instance, at position P in Fig. 10.a), such an observer will see three portions of the same front W(SV) passing through him at different times .

It must be stressed that the special feature described above is valid for the fiberglass epoxy composite used in this numerical example. The existence of this special behavior is not a common characteristic of all transversely isotropic media and depends on the numerical values of the elastic constants , so such features must be analyzed and determined for each specific material.

CONCLUSIONS

The problem of the geometry of propagation of waves in transversely isotropic media was studied. The solution of the homogeneous problem of motion for a fiberglass epoxy composite was analyzed and the numerical values of the resulta were presented.

The solution of the homogeneous problem of motion in an infinite medium provided a description of the behavior of plane stress waves in terms of velocities and paths of propagation . Using these concepts, a graphical method for the construction of the wave fronts was introduced .

As described above,the following conclusions can be stated:

(1) The graphical technique is found useful in the construction and the physical interpretation of the wave surfaces . The technique was applied to transversely isotropic materials but the concept can be extended to other cases having anisotropic symmetry.

(2) The wave surfaces corresponding to a fiberglass epoxy composite material were constructed .Special features were found in the W(SV) surface.

(3) It was found that energy carried by the SV waves in the fiberglass epoxy composite material is concentrated along certain preferential directions. This phenomenon can be explained in terms of the spreading geometry of plane wave segments .

The conclusions above suggest some possible applications and recommendations for ultrasonic NDE of transversely isotropic materials . For instance, the special features of the wave

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surface W(SV) for the fiberglass epoxy material can be used as an auxiliary criterion for the placement of transducers when experiments are designed . For example, the locations where the energy is focused (angle B'OD' in Fig.10) can be selected as the receiving region since the signals within this region are expected to be stronger due to larger displacement amplitudes. Also, certain locations in the medium can be selected as receiving points where up to five different wave front arrivals may occur, when the three modes of propagation are considered.

The knowledge of the geometry of the spreading can also be used for crack detection since the number of expected arrivals can be computed for each direction in the medium. Thus, the absence of the arrival of one of the expected portions of a wave front might indicate that an obstacle was in the path of that portion of the front. Such an obstacle might indicate the presence of a flaw.

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Table 1. Designation of Velocity Surfaces and Modes of Vibration
at Intersections with Coordinate Axes.

Propagation Direction	Particle Displacement Directions		
	x	y	z
x	V(P)	V(SH)	V(SV)
y	V(SH)	V(P)	V(SV)
z	V(SV)	V(SH)	V(P)

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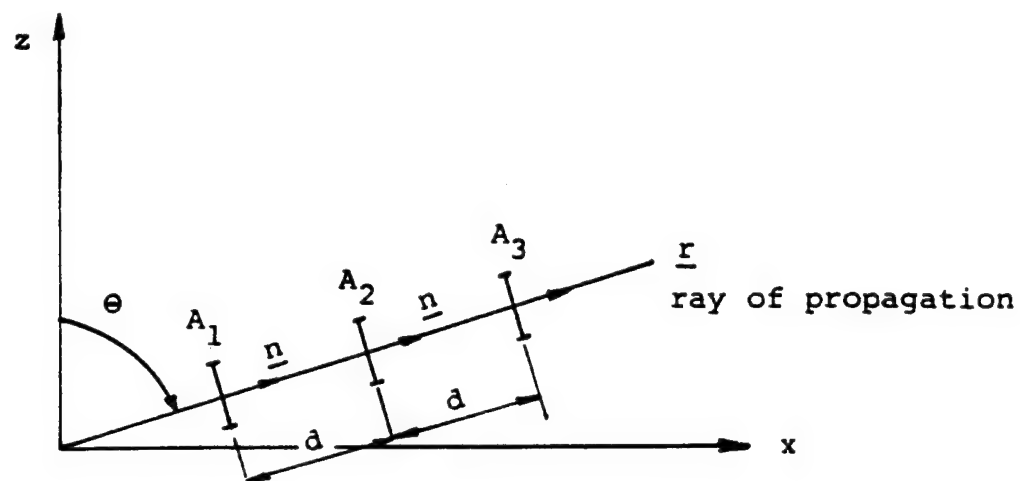
a) Isotropic medium, $v_n = d/\triangle t$
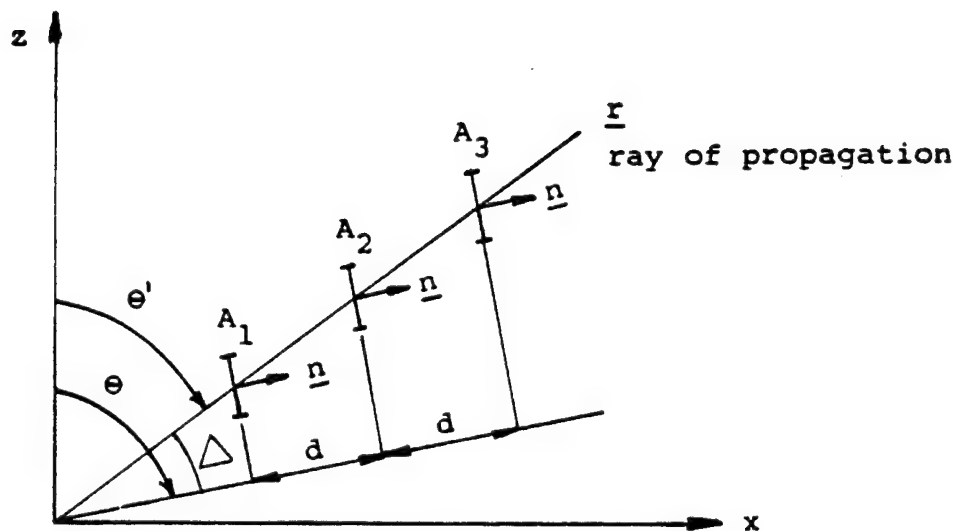
b) Anisotropic medium, $v_n = d/\triangle t$

Fig.1   Phase   velocity definition for a) isotropic   and   b)
        anisotropic   media   .   (A_1, A_2,   and A_3
        represent positions   of   wave   fronts
        in times t, t+$\triangle$t , t+2$\triangle$t.   For d= λ , $\triangle$t= T,
        v_n= λ /T = ω/k for periodic waves.)

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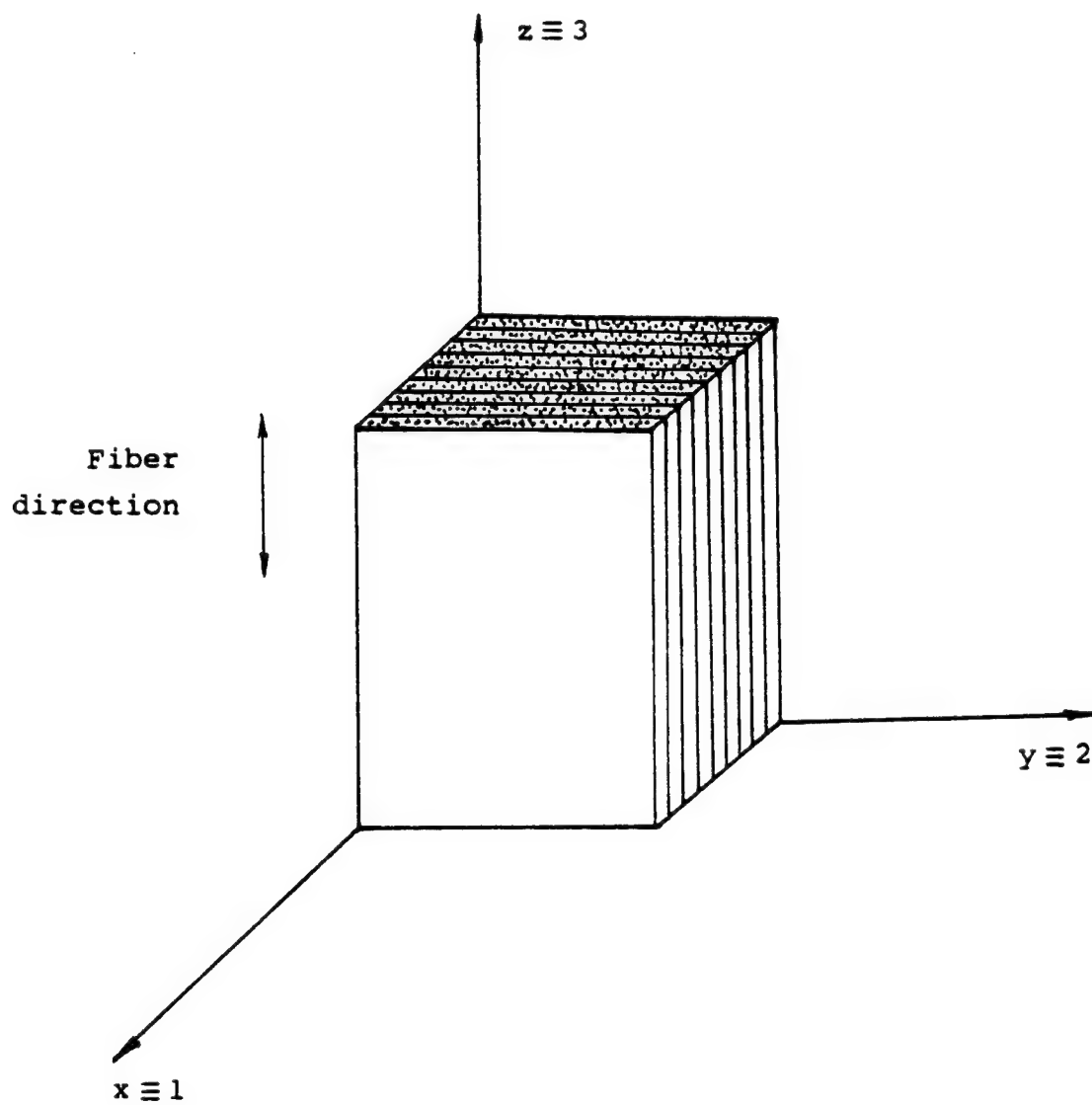
Fig.2   Position of material axes with respect to coordinate
        axes   for transversely isotropic medium.
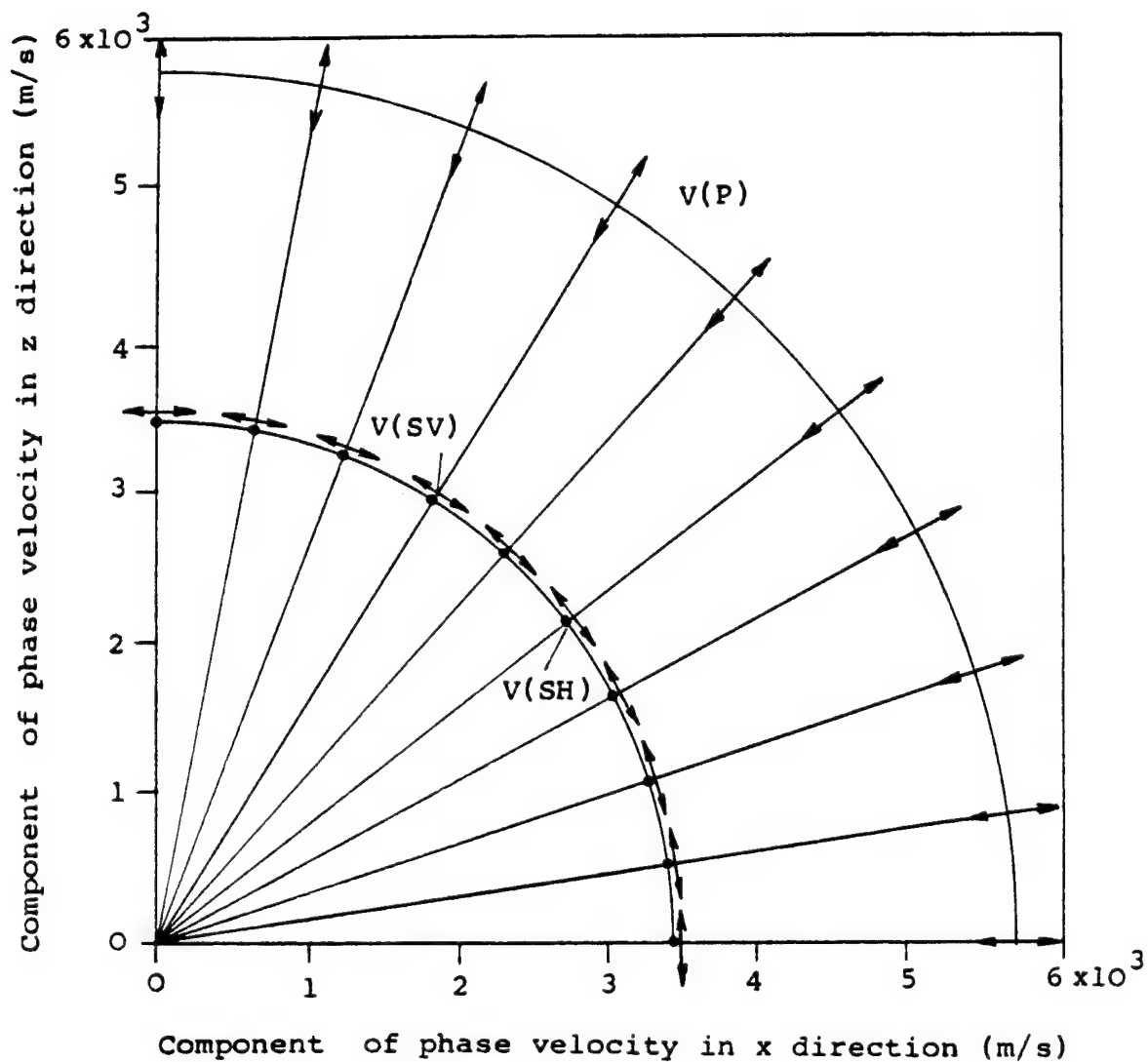
Fig.3   Phase velocity polar diagram for isotropic E glass
         fiber   material for positive x-z quadrant. (Arrows
         represent directions of particle displacements.)
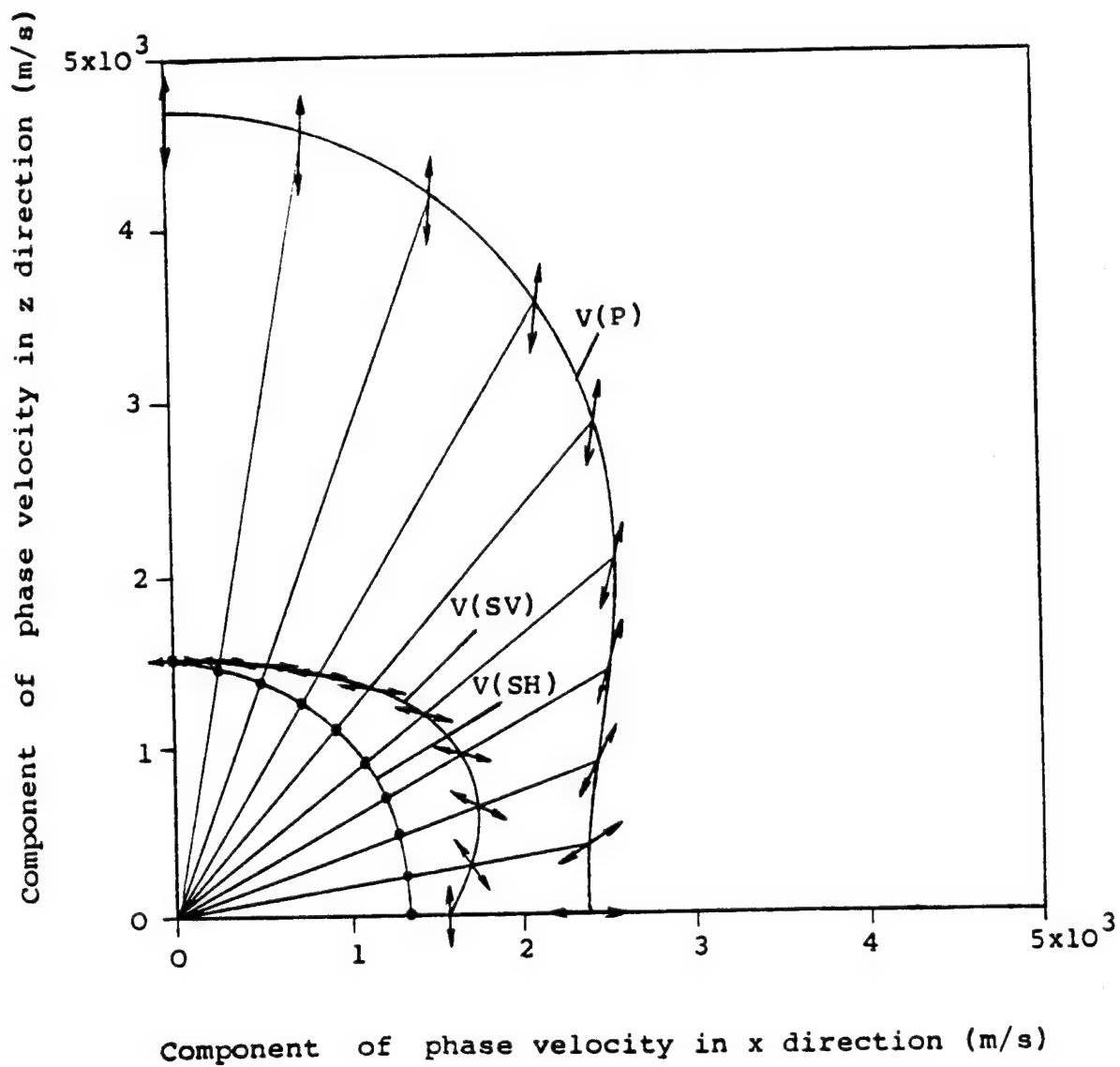
Fig.4   Phase velocity polar diagram for fiberglass epoxy composites for positive x − z quadrant. (Arrows represent direction of particle displacements.)
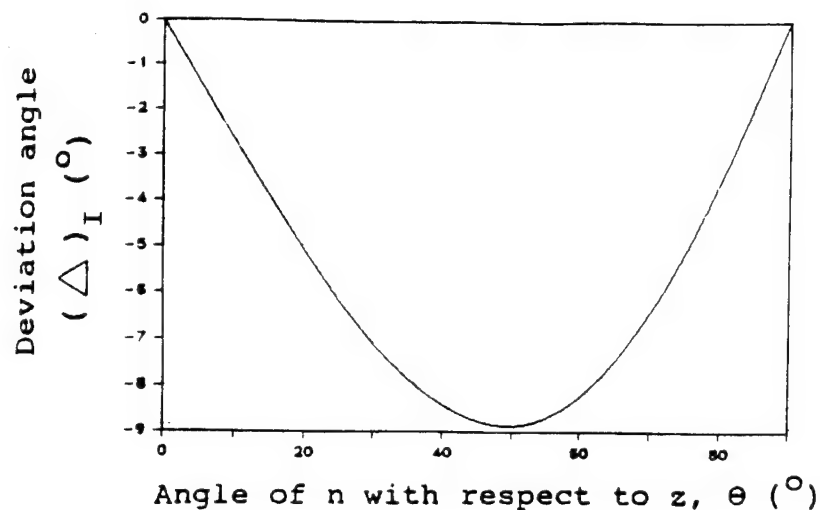
a) Deviation angles for V(SH)
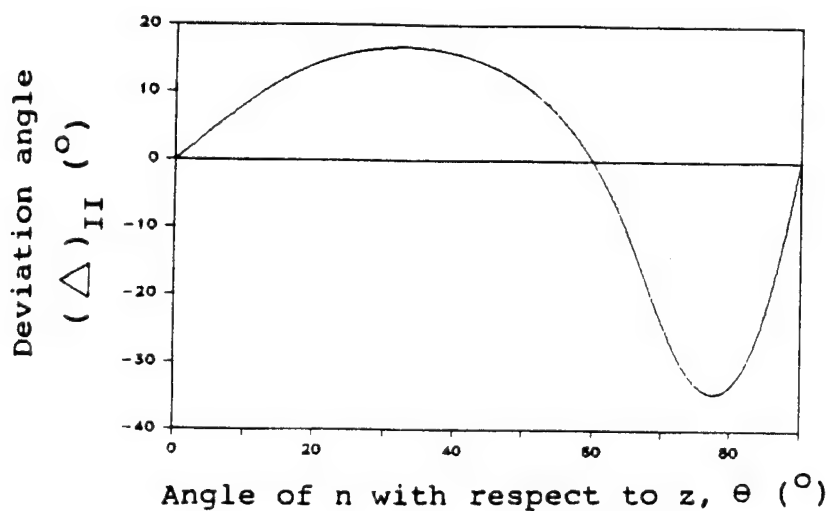
b) Deviation angles for V(SV)
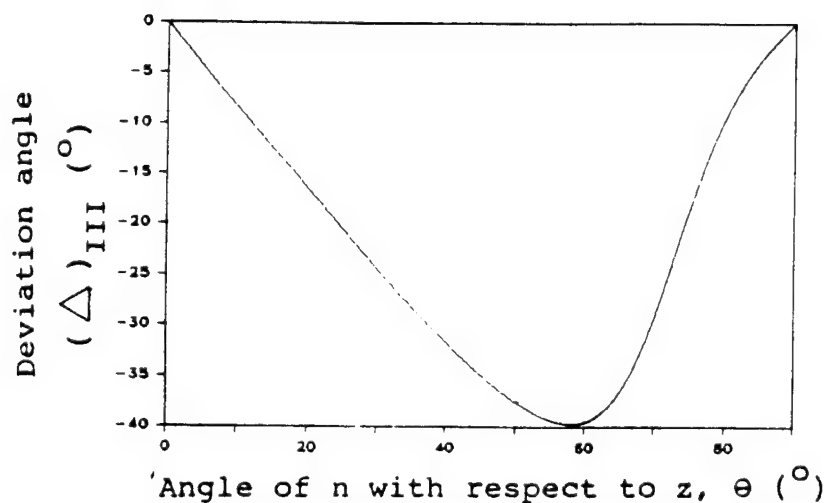
c) Deviation angles for V(P)

Fig.5   Deviation angles $(\triangle)_{I,II,III}$ between phase velocity and group velocity directions for phase velocities V(SH),V(SV) and V(P).
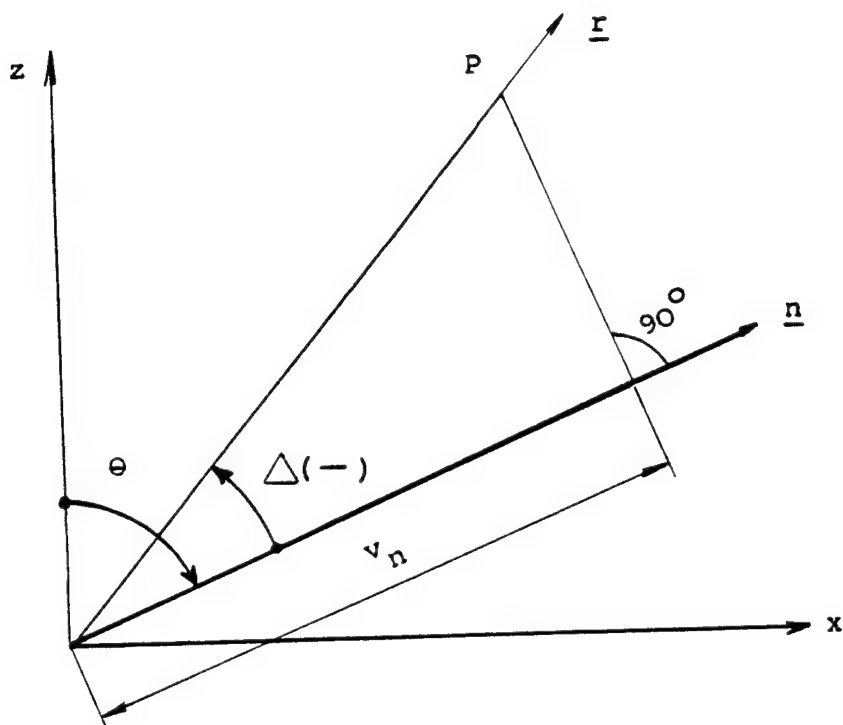
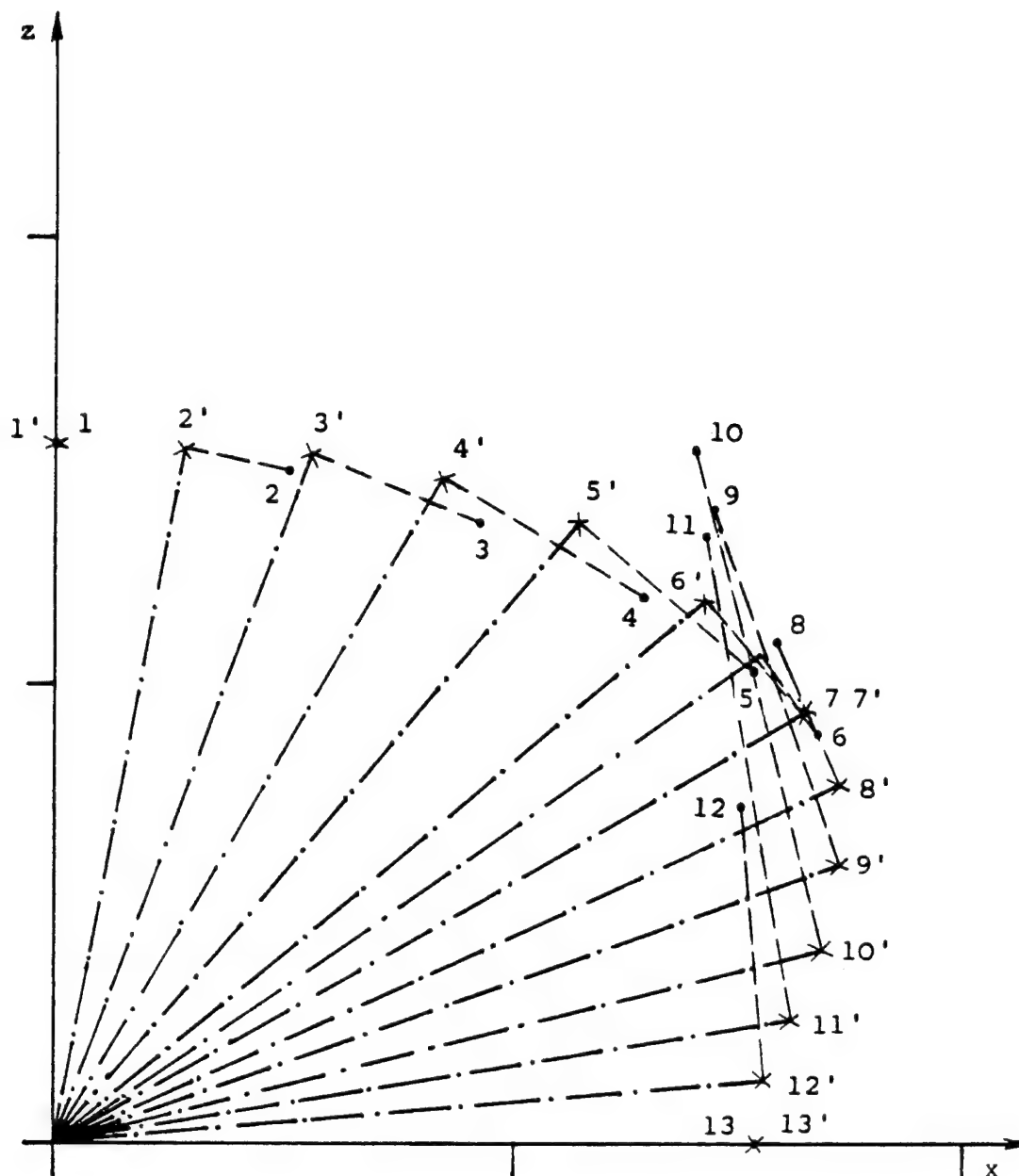Fig.6   Schematic for procedure for determination of points
         P on wave surface.

Fig.7' Construction of wave surface W(SV) from corresponding velocity surface V(SV) and deviation angles $(\triangle)_{II}$ for fiberglass epoxy composite , for positive x-z quadrant. (Points 1 through 13 are on wave surface : points 1'through 13' are on velocity surface.)
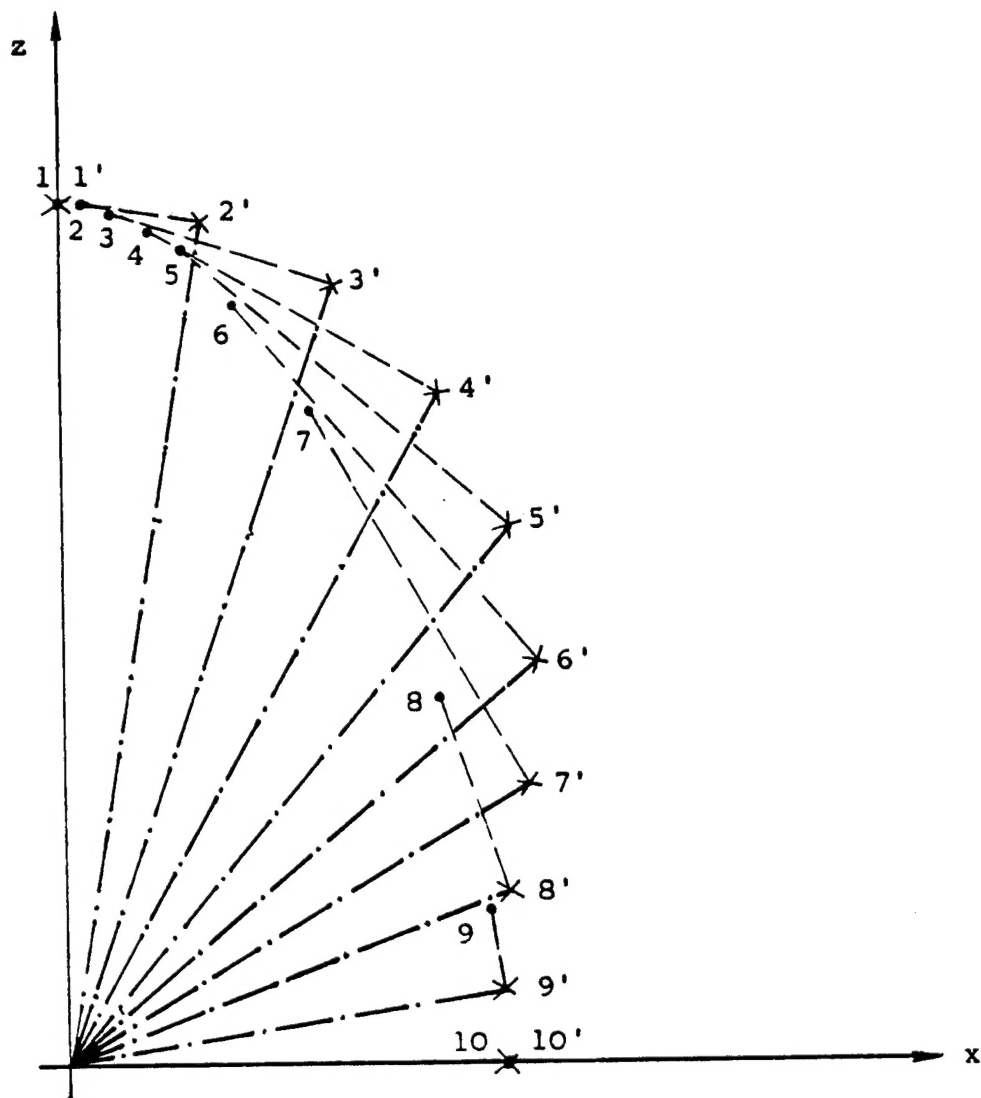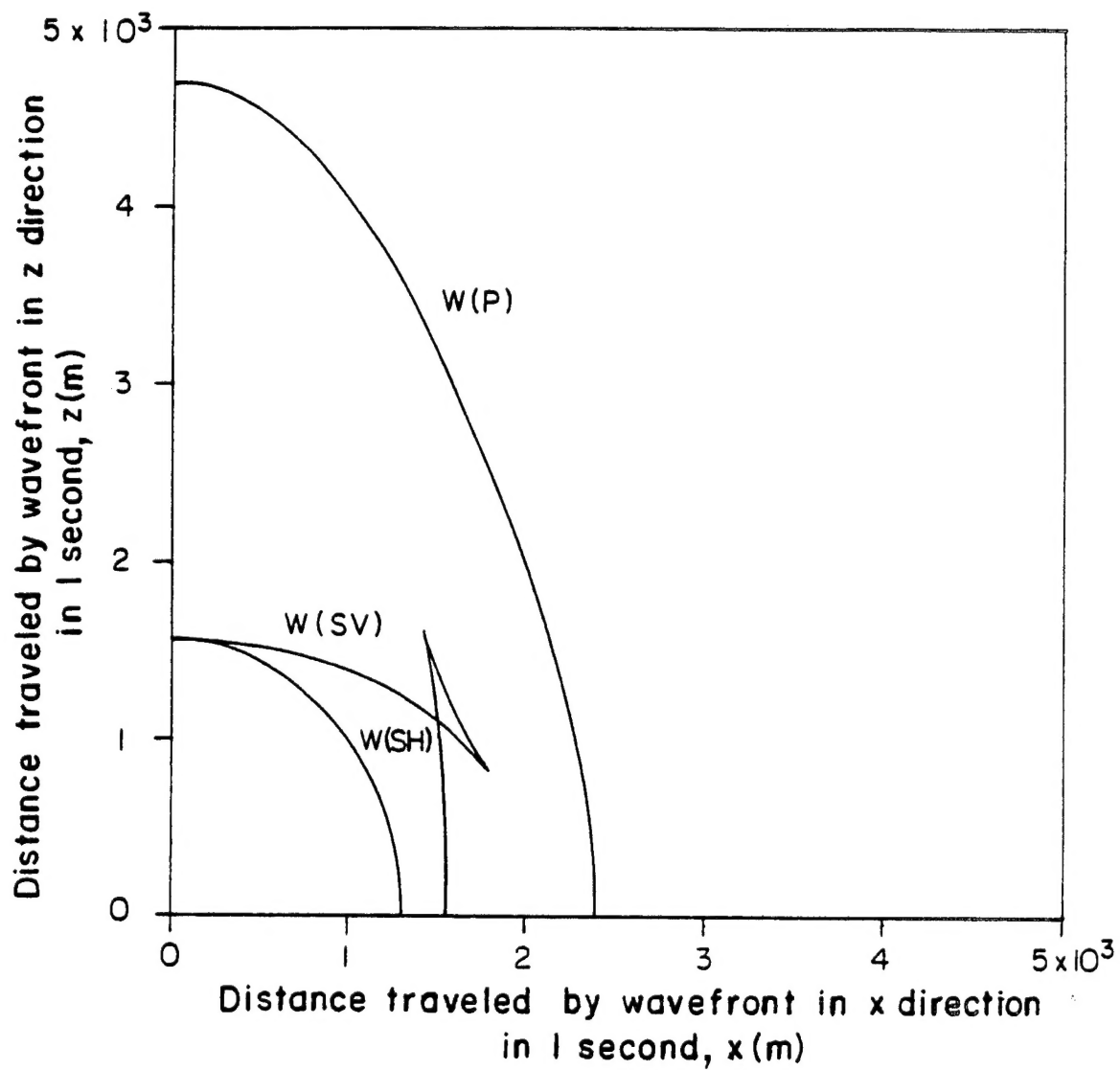
Fig.8 Construction of wave surface W(P) from corresponding velocity surface V(P) and deviation angles (△)ᵢᵢᵢ for fiberglass epoxy composite , for positive x-z quadrant. (Points 1 through 10 are on wave surface : points 1'through 10' are on velocity surface.)
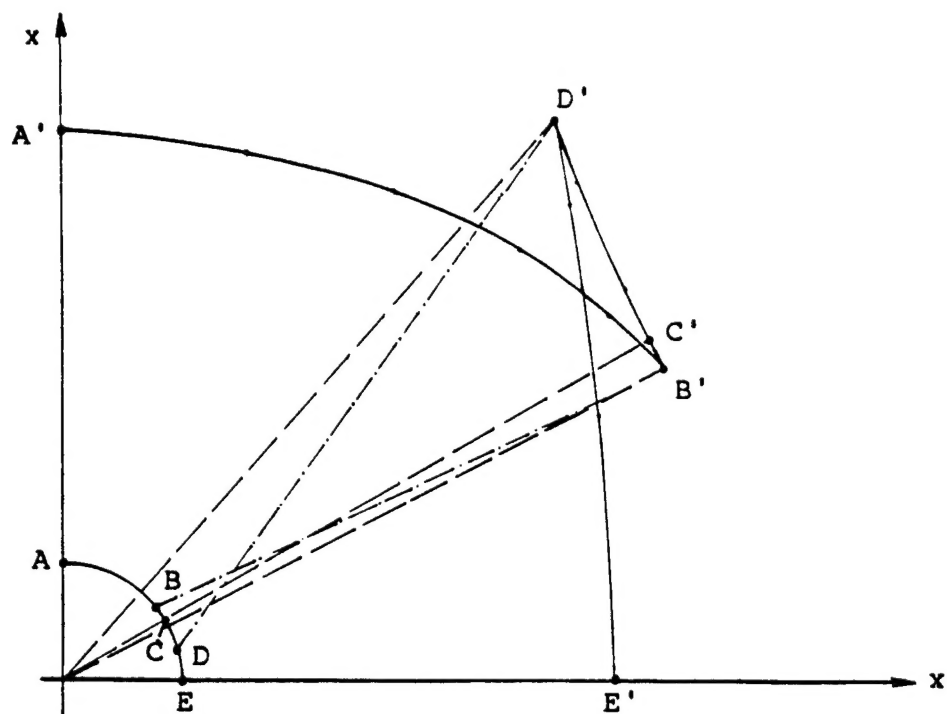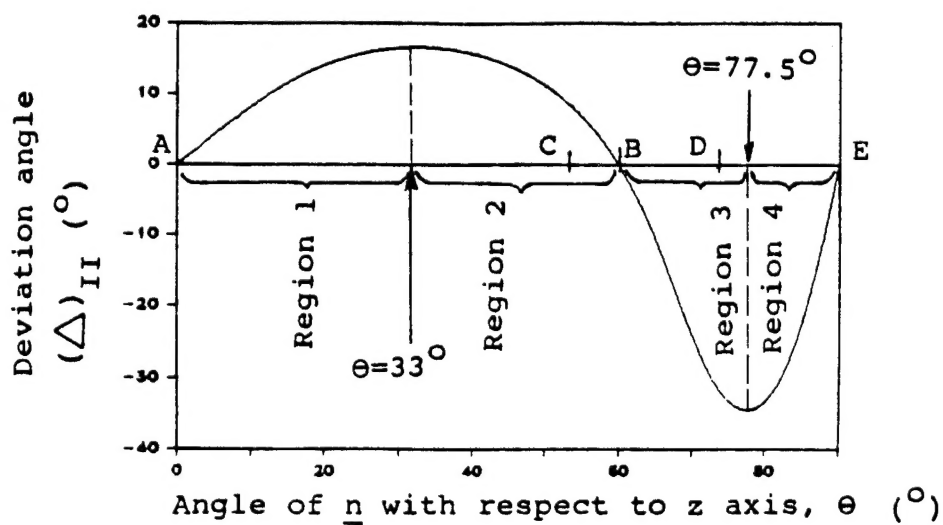
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Fig.9   Wave   surfaces   W(SH),   W(SV)   and   W(P)   for
        unidirectional   fiberglass   epoxy   composite   for
        positive   x-z   quadrant.

a) Wavefront assumed shapes for time $t = \Theta$ (ABCDE) and t=unity (A'B'C'D'E').

b), Deviation angles $(\triangle)_{II}$ showing regions and characteristic points.

Fig.10 Model of geometrical spreading for SV wave.

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16. Abstract

The homogeneous problem of stress wave propagation in unbounded transversely isotropic media is analyzed. By adopting plane wave solutions, the conditions for the existence of the solution are established in terms of phase velocities and directions of particle displacements. Dispersion relations and group velocities are derived from the phase velocity expressions. The deviation angles (e.g., angles between the normals to the adopted plane waves and the actual directions of their propagation) are numerically determined for a specific fiberglass epoxy composite. A graphical method is introduced for the construction of the wave surfaces using magnitudes of phase velocities and deviation angles. The results for the case of isotropic media are shown to be contained in the solutions for the transversely isotropic media.

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